THE LAPLACE–BELTRAMI OPERATOR ON RANK ONE SEMISIMPLE SYMMETRIC SPACES IN POLAR COORDINATES [1]

A. A. Artemov, V. F. Molchanov

G. R. Derzhavin Tambov State University, Russia

Let $\mathcal{X} = G/H$ be a semisimple symmetric space of rank one. The algebra of G-invariant differential operators on $\mathcal{X}$ is generated by the Laplace-Beltrami operator Δ corresponding to a G-invariant metric. It is very important (for many purposes) to know explicit expressions of Δ in various coordinate systems. For example, polar coordinates associated with the Cartan-Berger decomposition $G = HAK$ (for definitions, see Section 1) are necessary for the study of canonical and boundary representations, for the study of Poisson and Fourier transforms etc.

But, as we know, explicit expressions of Δ in polar coordinates are written in particular cases only.

For Riemannian (noncompact) symmetric space of rank one $\mathcal{X} = G/K$, the Laplace-Beltrami operator in polar coordinates has the form (see, for example, [4]):

$$\Delta = \frac{1}{A}\frac{\partial}{\partial r} A \frac{\partial}{\partial r} + L_S.$$

Here r is the distance between a point $x \in \mathcal{X}$ and the initial point x^0, S the sphere in $\mathcal{X}$ with center x^0 and radius r, the area A of S is given by

$$A = C \cdot \Big\{ \sinh(cr) \Big\}^{r_1} \Big\{ \sinh(2cr) \Big\}^{r_2}$$

where c is a number (written explicitly), r_1, r_2 are multiplicities of roots α, 2α, respectively, and, finally, L_S is the Laplace-Beltrami operator on S.

For real hyperbolic spaces (hyperboloids) $\mathcal{X} = G/H$, where $G = \mathrm{SO}_0(p,q)$, $H = \mathrm{SO}_0(p,q-1)$, the Laplace-Beltrami operator in polar coordinates is written as follows (see, for example, [1]). The hyperboloid $\mathcal{X}$ is a manifold in $\mathbb{R}^n$, $n = p+q$, defined by equation

$$-x_1^2 - \ldots, -x_p^2 + x_{p+1}^2 + \ldots + x_n^2 = 1.$$

Let spheres $S_1 \subset \mathbb{R}^p$ and $S_2 \subset \mathbb{R}^q$ are defined by equations $u_1^2 + \ldots + u_p^2 = 1$ and $v_1^2 + \ldots + v_q^2 = 1$, respectively. Polar coordinates t, u, v ($t \in \mathbb{R}$, $u \in S_1$, $v \in S_2$) in $\mathcal{X}$ are introduced by

$$x = (\sinh t \cdot u, \cosh t \cdot v).$$

Then

$$\Delta = \frac{1}{v}\frac{\partial}{\partial t} v \frac{\partial}{\partial t} + \frac{\Delta_1}{\sinh^2 t} - \frac{\Delta_2}{\cosh^2 t},$$

where

$$v = |\sinh t|^{p-1} (\cosh t)^{q-1},$$

Δ_1, Δ_2 the Laplace-Beltrami operators on S_1, S_2, respectively.

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A similar formula for Δ is true for hyperbolic spaces over complex numbers and octonions, see, for example, [3].

For arbitrary semisimple symmetric spaces $\mathcal{X} = G/H$ of rank one, the Laplace-Beltrami operator in polar coordinates t, s, where $t \in \mathbb{R}$, $s \in S$ (for S, see Section 2 below) was written in [5]:

$$\Delta = \frac{1}{v}\frac{\partial}{\partial t} v \frac{\partial}{\partial t} + \sum_j v_j(t) D_j,$$

where the function $v = v(t)$ is given by formula (2.14) below, $v_j(t)$ are some functions, D_j are some differential operators on S invariant with respect to K. But [5] does not contain explicit expressions of v_j and D_j. Apparently, such expressions seem to be rather complicated.

We succeeded in the obtaining explicit expressions of Δ in polar coordinates for all $\mathcal{X}$ – at points in a Cartan subset (i.e. points whose angular coordinates are equal to zero), see Theorem 2.1.

Notice that for para-Hermitian symmetric spaces of rank one, an explicit formula of Δ in horospherical coordinates is written in [2].

§1. Semisimple symmetric spaces of rank one

In this Section we recall some material from [5].

Let $\mathcal{X} = G/H$ be a semisimple symmetric space. It means that G is a connected semisimple Lie group, there is an involution $\sigma (\neq 1)$ of G such that H is an open subgroup in the subgroup G^σ of all points in G fixed under σ. We shall assume that G acts on $\mathcal{X}$ from the right and shall denote by $R(g) : x \mapsto xg$ the translation of $\mathcal{X}$ by g. Let us write x^0 for the initial point $\{H\}$ of $\mathcal{X}$.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H respectively. The involution σ of G gives rise to an involution σ of $\mathfrak{g}$ (we use the same symbol). The algebra $\mathfrak{g}$ can be written as the direct sum:

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$$

of $+1, -1$-eigenspaces of σ. The commutation relations are:

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \ [\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}, \ [\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h}.$$

The space $\mathfrak{q}$ can be identified with the tangent space of $\mathcal{X}$ at x^0.

There exists a Cartan involution τ of $\mathfrak{g}$ which commutes with σ. The algebra $\mathfrak{g}$ decomposes into the direct sum of $+1, -1$-eigenspaces for τ:

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p},$$

here $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$. The commutation relations are:

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \ [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \ [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

There is a joint decomposition:

$$\mathfrak{g} = \mathfrak{k} \cap \mathfrak{h} + \mathfrak{k} \cap \mathfrak{q} + \mathfrak{p} \cap \mathfrak{h} + \mathfrak{p} \cap \mathfrak{q}.$$

All these decompositions are orthogonal with respect to the Killing form $B_\mathfrak{g}$ of $\mathfrak{g}$.

We assume that $\mathfrak{p} \neq 0$ and $\mathfrak{p} \cap \mathfrak{h} \neq 0$, i. e. that $\mathfrak{g}$ and $\mathfrak{h}$ are non-compact (excluding in this way the Riemannian case). Then $\mathfrak{p} \cap \mathfrak{q} \neq 0$, $\mathfrak{k} \cap \mathfrak{q} \neq 0$.

From now on we assume that the rank of $\mathcal{X}$ is equal to 1. It means that the dimension of any Cartan subspace of $\mathfrak{q}$ (a maximal Abelian subalgebra of $\mathfrak{q}$ consisting of semisimple elements) is equal to 1. Fix such a subspace $\mathfrak{a}$ lying in $\mathfrak{p} \cap \mathfrak{q}$.

Let $A = \exp \mathfrak{a}$. The centralizer of the subgroup A in G is the product AM of two subgroups A and M whose intersection is the identity element e of G. The subgroup M is closed and reductive.

The algebra $\mathfrak{a}$ is splitted in $\mathfrak{g}$. The corresponding root decomposition is:

$$\mathfrak{g} = \mathfrak{g}_{-2\alpha} + \mathfrak{g}_{-\alpha} + \mathfrak{g}_0 + \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha},$$

where the root α is an element of $\mathfrak{a}^*$. The subspaces $\mathfrak{g}_{\pm 2\alpha}$ may sometimes be absent. The involutions σ and τ give isomorphisms $\mathfrak{g}_{j\alpha} \to \mathfrak{g}_{-j\alpha}$, $j = 0, \pm 1, \pm 2$. Let us denote:

$$r_j = \dim \mathfrak{g}_{j\alpha} = \dim \mathfrak{g}_{-j\alpha}, \quad j = 1, 2,$$

$$r = 1 + r_1 + r_2.$$

Set

$$\mathfrak{q}_j = (1-\sigma)\mathfrak{g}_{j\alpha}, \ \mathfrak{h}_j = (1+\sigma)\mathfrak{g}_{j\alpha}, \quad j = 1, 2.$$

Then $\dim\mathfrak{q}_j = \dim\mathfrak{h}_j = r_j$, the spaces $\mathfrak{q}$ and $\mathfrak{h}$ decompose into the direct orthogonal (with respect to the Killing form) sums:

$$\mathfrak{q} = \mathfrak{a} + \mathfrak{q}_1 + \mathfrak{q}_2, \quad \mathfrak{h} = \mathfrak{m} + \mathfrak{h}_1 + \mathfrak{h}_2,$$

where $\mathfrak{m} = \mathfrak{g}_0 \cap \mathfrak{h}$, so that $\mathfrak{g}_0 = \mathfrak{a} + \mathfrak{m}$. Therefore, $\dim\mathfrak{q} = r$. The algebra $\mathfrak{m}$ is the Lie algebra of M.

Fix a basis element $A_0 \in \mathfrak{a}$ so that

$$\alpha(A_0) = 1.$$

Then $\mathrm{ad}A_0 \cdot X = \pm jX$ for $X \in \mathfrak{g}_{\pm j\alpha}$, $j = 0, 1, 2$. Let us denote:

$$a_t = \exp tA_0. \tag{1.1}$$

It is more convenient for us to consider, instead of the Killing form $B_\mathfrak{g}$, the proportional form $\langle X, Y\rangle$ normalized by the condition

$$\langle A_0, A_0 \rangle = 1.$$

Recall, that the signature of a non-degenerate quadratic form on a vector space is the pair (p, q) indicating the number of plus and minus signs in a canonical expression of this form.

Denote the signatures of the form $\langle\, , \rangle$ on $\mathfrak{q}$ and $\mathfrak{q}_j$ by (r^+, r^-) and (r_j^+, r_j^-) respectively $(j = 1, 2)$. On $\mathfrak{a}$ its signature is $(1, 0)$. Therefore,

$$r^+ = 1 + r_1^+ + r_2^+, \quad r^- = r_1^- + r_2^-.$$

The signatures of $\langle\, , \rangle$ on $\mathfrak{h}_j$ are (r_j^-, r_j^+).

The operator $\mathrm{ad}A_0$ gives an isomorphism of $\mathfrak{q}_j$ onto $\mathfrak{h}_j$ and conversely and vanishes on $\mathfrak{g}_0 = \mathfrak{a} + \mathfrak{m}$. The subspaces $\mathfrak{q}_j$ and $\mathfrak{h}_j$ are eigenspaces for $(\mathrm{ad}A_0)^2$ with eigenvalues j^2. In particular, it implies the following.

Lemma 1.1. *Let $X \in \mathfrak{q}_j$, $j = 1, 2$. Then the element $Y = (1/j)\mathrm{ad}A_0 \cdot X$ belongs to $\mathfrak{h}_j$ and the operator $\mathrm{ad}A_0$ transforms the elements X, Y by the matrix:*

$$\begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix},$$

so that in this basis X, Y the operator $\mathrm{Ad}\, a_t$ has the matrix

$$\begin{pmatrix} \cosh jt & \sinh jt \\ \sinh jt & \cosh jt \end{pmatrix}.$$

The same is true for $X \in \mathfrak{h}_j$, *then* $Y \in \mathfrak{q}_j$.

Let P denote the orthogonal projection operator in $\mathfrak{g}$ onto $\mathfrak{q}_1 + \mathfrak{q}_2$. Lemma 1.1 gives:

Lemma 1.2. *If* $X \in \mathfrak{q}_j$, *then*

$$P(\mathrm{Ad}a_t \cdot X) = \cosh jt \cdot X,$$

and if $X \in \mathfrak{h}_j$, *then*

$$P(\mathrm{Ad}a_t \cdot X) = \frac{1}{j} \sinh jt \cdot \mathrm{ad}A_0 \cdot X.$$

Let K be the Lie subgroup of G with Lie algebra $\mathfrak{k}$. Then K is connected, closed and contains the centre of G. The involution τ can be lifted to G so that $K = G^\tau$. The group K is compact if and only if the centre of G is finite. Then K is a maximal compact subgroup of G.

§ 2. The Laplace-Beltrami operator

The bilinear form $\langle\, ,\, \rangle$ on $\mathfrak{q}$ gives rise to a G-invariant metric Q on $\mathcal{X}$:

$$Q_{x^0 g}(dR(g)_{x^0}L,\ dR(g)_{x^0}M) = \langle L, M\rangle \tag{2.1}$$

where $L, M \in \mathfrak{q}$. Let Δ be the Laplace-Beltrami operator on $\mathcal{X}$ generated by this metric. Let us recall that the Laplace-Beltrami operator Δ on a manifold generated by a metric $\sum g_{ij}(x)dx_i dx_j$ is defined as follows:

$$\Delta = \frac{1}{\sqrt{\overline{g}}} \sum_j \frac{\partial}{\partial x_j} \sum_i g^{ij} \sqrt{\overline{g}} \frac{\partial}{\partial x_i}, \tag{2.2}$$

where (g^{ij}) is the inverse matrix of (g_{ij}) and $\overline{g} = |\det(g_{ij})|$.

Let us introduce a system of polar coordinates on $\mathcal{X}$ – by means of the Cartan-Berger decomposition $G = HAK$. This decomposition gives that any $x \in \mathcal{X}$ can be written in the form

$$x = x^0 a_t k, \tag{2.3}$$

where a_t is given by (1.1) and $k \in K$. If $t \neq 0$, then the element k in (2.3) is defined up to the multiplication from the left by elements in $K \cap H \cap M$, so that the manifold $A \times S$, where $S = K/K \cap H \cap M$, is mapped in the natural way onto $\mathcal{X}$. The tangent space to S at the initial point can be identified with the direct sum of spaces

$$\mathfrak{k} \cap \mathfrak{h}_1,\ \mathfrak{k} \cap \mathfrak{q}_1,\ \mathfrak{k} \cap \mathfrak{h}_2,\ \mathfrak{k} \cap \mathfrak{q}_2. \tag{2.4}$$

Let us take orthogonal bases X_i, $\langle X_i, X_i\rangle = -1$, in these spaces, here $1 \leqslant i \leqslant r_1^+$, $r_1^+ + 1 \leqslant i \leqslant r_1$, $r_1 + 1 \leqslant i \leqslant r_1 + r_2^+$, $r_1 + r_2^+ + 1 \leqslant i \leqslant r_1 + r_2 = r - 1$, respectively. We introduce local coordinates $t, u_1, ..., u_{r-1}$ on $\mathcal{X}$ (polar coordinates) by

$$x = x^0 g, \tag{2.5}$$

where

$$g = a_t \exp \sum_{i=1}^{r-1} u_i X_i. \tag{2.6}$$

Theorem 2.1. *At points $x = x^0 a_t$ the Laplace-Beltrami operator Δ reads:*

$$\Delta = \frac{\partial^2}{\partial t^2} + (r_1^+ \coth t + r_1^- \tanh t + 2r_2^+ \coth 2t + 2r_2^- \tanh 2t)\frac{\partial}{\partial t} +$$

$$+\frac{1}{\sinh^2 t}\Delta_1^+ - \frac{1}{\cosh^2 t}\Delta_1^- + \frac{1}{\sinh^2 2t}\Delta_2^+ - \frac{1}{\cosh^2 2t}\Delta_2^-,$$

where Δ_1^+, Δ_1^-, Δ_2^+, Δ_2^- *are the usual Laplace operators* ($\sum \partial^2/\partial u_j^2$) *in spaces (2.4) respectively.*

The theorem implies from the following two lemmas.

Lemma 2.2. *In the coordinates $t, u_1, ..., u_{r-1}$ the metric Q_x has the form*

$$Q_x = dt^2 + \sum b_{ij}(t,u) du_i du_j, \tag{2.7}$$

with

$$\frac{\partial}{\partial u_i}\Big|_{u=0} b_{mm} = 0, \tag{2.8}$$

$$b_{lm}\big|_{u=0} = 0, \quad l \neq m. \tag{2.9}$$

Proof. Let us consider tangent vectors

$$\frac{\partial}{\partial t}, \frac{\partial}{\partial u_1}, ..., \frac{\partial}{\partial u_{r-1}}$$

at a point x given by (2.5), (2.6). By (2.1), they are the images of some vectors $T, U_1, \ldots, U_{r-1}$ in $\mathfrak{q}$ under the map $dR(g)_{x^0}$. Clearly that $T = A_0$, i.e.

$$\frac{\partial}{\partial t} = dR(g)_{x^0} A_0.$$

Let us determine $U_m \in \mathfrak{q}$. Let $\gamma(\mu)$ be a curve in G such that $\gamma(0) = e$ and $\gamma'(0) = U_m$. Then the following condition has to be satisfied:

$$x^0 \gamma(\mu) g = x^0 a_t \exp\Big(\mu X_m + \sum u_i X_i\Big).$$

From here and (2.6) we have

$$x^0 \gamma(\mu) = x^0 a_t \exp\Big(\mu X_m + \sum u_i X_i\Big) \exp\Big(-\sum u_i X_i\Big) a_t^{-1}.$$

Let us apply to the product of these exponents the Taylor expansion of the product in canonical (logarithmic) coordinates up to the order two and differentiate the obtained equality with respect to μ at $\mu = 0$. Then we obtain:

$$U_m = P\big(\mathrm{Ad}\, a_t \cdot \{X_m - \frac{1}{2}\sum_{i\neq m} u_i [X_m, X_i] + ...\}\big), \tag{2.10}$$

where dots mean terms of the order greater than one in u_i.

Since the vectors X_i belong to spaces (2.4), the vector U_m lies in $\mathfrak{q}_1 + \mathfrak{q}_2$. Therefore, $\langle A_0, U_m \rangle = 0$ and $Q_x(\partial/\partial t, \partial/\partial u_m) = 0$. It proves (2.7).

In order to prove (2.8) we have to show that linear terms (in u_i) in $\langle U_m, U_m \rangle$ vanish. We use Lemma 1.2. It follows from (2.10) that the linear function u_i enters $\langle U_m, U_m \rangle$ with a coefficient that is the product of

$$\langle X_m, [X_m, X_i] \rangle \quad \text{or} \quad \langle \mathrm{ad} A_0 \cdot X_m, [X_m, X_i] \rangle \tag{2.11}$$

and a function of t. Both inner products in (2.11) are equal to zero: the first one is equal to $\langle [X_m, X_m], X_i \rangle = 0$, the second one vanishes because $\mathrm{ad}A_0 \cdot X_m$ belongs to $\mathfrak{p} \cap \mathfrak{q}$ and is orthogonal to $[X_m, X_i] \in \mathfrak{k}$.

Finally, let us prove (2.9). For that, we have to show that $\langle U_l, U_m \rangle = 0$ when $l \neq m$ and $u = 0$. From (2.10) we have for $u = 0$:

$$\langle U_l, U_m \rangle = \langle P(\mathrm{Ad}a_t \cdot X_l),\ P(\mathrm{Ad}a_t \cdot X_m) \rangle. \tag{2.12}$$

The right hand side is equal to $\langle X_l, X_m \rangle$ or $\langle \mathrm{Ad}A_0 \cdot X_l,\ X_M \rangle$ with a coefficient depending on t. If $l \neq m$, then these inner products are equal to zero: the first one because of orthogonality of the bases X_i, the second one because $\mathrm{ad}A_0 \cdot X_l \in \mathfrak{p} \cap \mathfrak{q}$ and $X_m \in \mathfrak{k}$. □

Denote

$$v = \sqrt{|\det(b_{ij})|}$$

Lemma 2.3. *At points $x = x^0 a_t$, $t \neq 0$, the metric Q_x has the form*

$$Q_x = dt^2 + \sinh^2 t \cdot \sum du_i^2 - \cosh^2 t \cdot \sum du_i^2 + \sinh^2 2t \cdot \sum du_i^2 - \cosh^2 2t \cdot \sum du_i^2, \tag{2.13}$$

where i ranges sets mentioned above, so that

$$v = (\cosh t)^{r_1^+} \cdot |\sinh t|^{r_1^-} \cdot (\cosh 2t)^{r_2^+} \cdot |\sinh 2t|^{r_2^-}. \tag{2.14}$$

Proof. It follows from (2.9) that the matrix (b_{ij}) at $x^0 a_t$ is diagonal. By (2.12), a diagonal entry b_{mm} is equal to

$$\langle U_m, U_m \rangle = \langle P(\mathrm{Ad}a_t \cdot X_m),\ P(\mathrm{Ad}a_t \cdot X_m) \rangle.$$

Let $X_m \in \mathfrak{k} \cap \mathfrak{q}_j$, then by Lemma 2.2, $P(\mathrm{Ad}a_t \cdot X_m) = \cosh jt \cdot X_m$, so that

$$\langle U_m, U_m \rangle = -\cosh^2 jt.$$

Let $X_m \in \mathfrak{k} \cap \mathfrak{h}_j$, then $P(\mathrm{Ad}a_t \cdot X_m) = \sinh jt \cdot (1/j)\mathrm{ad}A_0 \cdot X_m$, so that

$$\langle U_m, U_m \rangle = \frac{1}{j^2}\sinh^2 jt \cdot \langle \mathrm{ad}A_0 \cdot X_m, \mathrm{ad}A_0 \cdot X_m \rangle = -\sinh^2 jt \cdot \langle X_m, X_m \rangle = \sinh^2 jt.$$

It proves (2.13) and, therefore, (2.14). □

Now we can finish the proof of Theorem 2.1.

Let (b^{ij}) be the inverse matrix of (b_{ij}). It follows from (2.8) and (2.9) that

$$\frac{\partial}{\partial u_j}\Big|_{u=0} b^{ij} v = 0.$$

Therefore, by (2.2), we have at points $x = x^0 a_t$:

$$\Delta = \frac{1}{v}\frac{\partial}{\partial t} v \frac{\partial}{\partial t} + \sum \frac{1}{b_{ii}} \frac{\partial^2}{\partial u_i^2}.$$

It remains to substitute values of v and b_{ii} at $x^0 a_t$, see (2.13), (2.14).

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